The Parabolic Reflector as an Acoustical Amplifier

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The history of the parabolic reflector as an acoustical amplifier is described, followed by an analysis of the theoretical amplification under various conditions. The complete theory is presented in an appendix. Measurements carried out confirm the theory. Examples illustrate the use of the reflector and methods to improve its performance. The application of the parabolic reflector for stereophonic recordings is discussed.

0 INTRODUCTION

The parabolic reflector, or parabola for short, is, of course, no major chapter in acoustical science. It poses, however, interesting and well-defined problems, both theoretical and practical. We hope that some of the questions may be answered by this paper and that it can serve as a reference for future work within the field of biological acoustics. Over the years we discussed different aspects with numerous persons—biologists, amateur recordists, and scientists. Nobody had as complete a picture as is presented here. It is the best knowledge available to date, unless someone comes up with hitherto hidden information. We think that the historical side in particular could contain surprises. Comments are welcome.

1 SOME HISTORICAL NOTES

Most textbooks on acoustics contain some words about parabolic reflectors for acoustical purposes. The concentration and behavior of sound at curved surfaces has long been a puzzle, and many explanations have been offered. J. W. S. Rayleigh [1] devotes several pages to these phenomena, citing among other things the whispering gallery in St. Paul's Cathedral in London and the Ear of Dionysius in Syracuse, Sicily. The acoustical background of these is further treated and explained by the great acoustical pioneer W. C. Sabine [2]. The use of parabolic shells in churches is described in Cremer [3]. These are just a few publications in the acoustical literature, but none contains any quantitative analysis. Although the fact that curved surfaces can concentrate sound under specific conditions has been known for a long time, the physics behind this phenomenon was not explained until the introduction of geometric acoustics, which uses the analogy with light beams and their reflections (Figs. 1 and 2).

While interest in this reflection of sound has in the past mainly been roused by curiosity, practical applications of parabolic reflectors started around 1930. Two papers published in the United States date from that time. Olson and Wolff [4] suggested the use of a parabolic construction working as horn at low frequencies and as reflector at higher frequencies. This construction of H. F. Olson was obviously less successful than his later microphone inventions. In the second paper Hanson [5] reported the use of parabolic reflectors in connection with radio programs.

The theory behind the described instruments is,

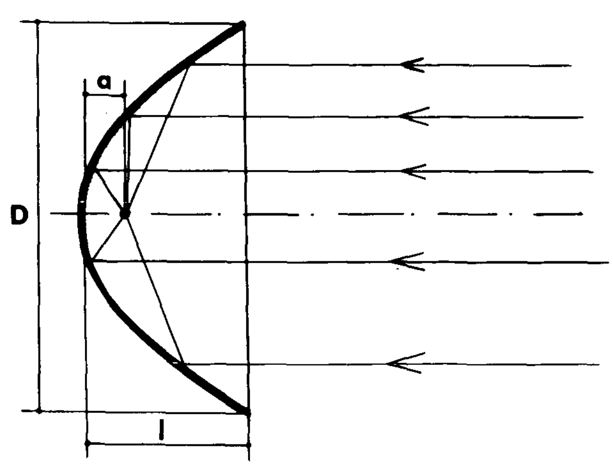

Fig. 1. The parabolic reflector can be used to concentrate sounds. Its function is easily understood with the help of geometrical acoustics.

however, very meager. With the advances in microphone design later in the 1930s, the use of the bulky parabolic microphones for professional purposes faded. In one field, however, parabolic reflectors have been used ever since, namely, when recording wildlife sounds. There are three major reasons for this. Wildlife sounds are in general very weak and need amplification without increasing the electronic noise. They are fairly high pitched and thus can be concentrated with reflectors of reasonable dimensions. Finally, through the directivity of the reflector it is possible to pick up one soloist from a chorus without coming too close and thus disturbing the wildlife. These factors were not fully understood when the first experiments took place at Cornell University in Ithaca, NY. I quote from the *Newsletter to Members of the Laboratory of Ornithology*, issue 24, April 1962, where Professor Peter Paul Kellogg (Fig. 3) writes about this historical event:

> During the winter of 1931–1932 Keane and I, then an instructor of ornithology and a registered graduate student, developed an idea for a parabola for biological sound-recording (in cooperation with Dr. Hugh Reed, then head of Zoology, and Professor Harley Howe of the Physics Department). Because a parabolic reflector had a known tendency to discriminate against low frequencies, our idea was not enthusiastically accepted at first, but we plunged ahead with our experiments on size, focal length, weight, construction, etc. In May of 1932 this crude, awkward, bulky "dish" was first set up in the field to record the voice of the Yellow-breasted Chat. Results? This particular Chat recording was used as a "standard of excellence" of recorded bird sound for a long while thereafter. Since then the parabola itself has proved one of the most valuable pieces of field equipment in use for biological recording.

As an aside it might be added that the recording was made optically on the sound track of a film, which at that time was the common method of making wildlife recordings at the Laboratory of Ornithology.

It is quite clear that before making his experiments, Kellogg knew of the two papers mentioned earlier. The reflector constructed by Kellogg was 32 in (0.81 m) in diameter, and this size has been in use ever since. Sometimes smaller reflectors, with diameters down to 0.5 m, have also been used.

In a subsequent paper [6] Kellogg described some physical properties of his reflector. In a diagram he gives the amplification, but no mention is made of any theoretical deduction from the results. During the 1930s the measuring technique was in its infancy, and the diagram that Kellogg shows in his 1938 paper is only a sketch.

The literature where today most descriptions of the parabolic reflector can be found is consequently confined to handbooks and papers concerned with wildlife sound recording. Most texts are limited to treating the directivity of the reflector, and nowhere are theoretical deductions made.

The theory of the acoustical behavior of the parabolic reflector is given as an appendix to this paper. It is an evaluation of the basic works by Rocard [7] and Gutin [8] published in the 1930s. These two papers were most certainly based on military interest in the field. At that time, sound sources such as airplanes could be spotted at night only by listening devices using stereophonic hearing with extreme bases or highly directional microphones. The technique of acoustical localization of airplanes became obsolete when the planes attained higher speeds. But even during the last world war the systems were still in use in air defense.

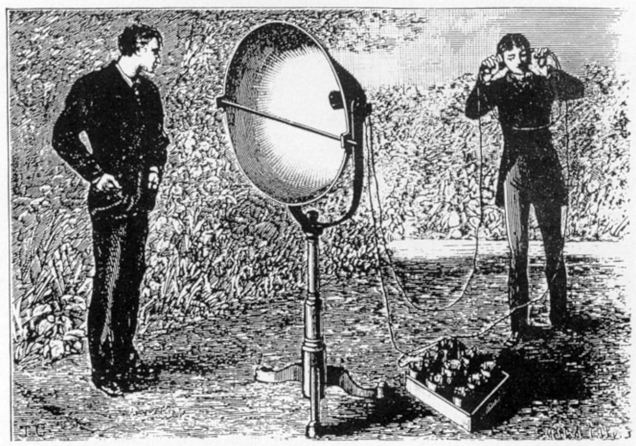

Fig. 2. This is not a parabolic reflector for acoustical purposes. The picture, circulated in the press in 1881, could wrongly be attributed to sounds. It shows the Photophone, an invention published by Alexander Graham Bell and Sumner Tainter on 1880 April 26, which was used to transmit sound by light beams. The receiver had a selenium photocell at the focus. It is, however, a fair guess that the inventors noticed the acoustical properties of the reflector during their experiments. The microphone, which had been invented by that time, could of course have been used as a receiver. It is rather surprising that no description of this combination can be found in the literature.

Fig. 3. Professor Peter Paul Kellogg in 1962, almost certainly the first person to use the parabolic reflector for recording wildlife sounds. The first recording, made in 1932 May, was of a bird, the Yellow-breasted Chat, *Icteria virens*. It was for a long time used as standard of excellence.

The theory presented in the Appendix was developed using computer-calculated diagrams and modern measurements and applying today's terminology. The measurements prove the theory nicely, and we hope that the deductions and measurements will improve the understanding of the parabolic reflector as a useful tool for wildlife sound studies.

Our own interest in the parabolic reflector goes back to the 1950s, when the first battery-operated tape recorders appeared and made wildlife sound recording easier (Fig. 4).

2 THEORETICAL AMPLIFICATION OF THE PARABOLIC REFLECTOR

The full theory is given in the Appendix, including equations referred to in the following.

The amplification, at focus, of sounds parallel to the axis is given by Eq. (14) and diagrammed in Fig. 5 for three of the reflector shapes illustrated in Fig. 6. The amplification of the reflector increases by 6 dB per octave over the frequency region where the focal distance is of the same order as the sound wavelength, $a/\lambda > 1$. For low frequencies, $a/\lambda < 1/64$, the amplification is 0 dB. In the interval $1/64 < a/\lambda < 1$ the amplification varies according to the sinus term in Eq. (14). It is the interaction between the direct sound and the reflected sound that causes a standing-wave pattern in front of the reflector when these components are nearly equal in strength. Compare with reflections at a plane surface. This effect is more pronounced for flat reflectors, where l/a is less than 1.

Omitting the sinus term will give a simple series of straight lines, 6 dB per octave, starting at different frequencies depending on the ratio l/a (Fig. 7). For reflectors with the same diameter D, but with different shapes (Fig. 6), the relative amplification at a fixed value of $a/\lambda >> 1$ can be seen from Fig. 8. The most efficient reflector is consequently one with $l/a = 4$. The variation is, however, rather small, a few decibels. For practical reasons $l/a = 1$ is the most widely used shape. For such a reflector it is easy to locate the focus in the plane of the mouth, and it is not too bulky to

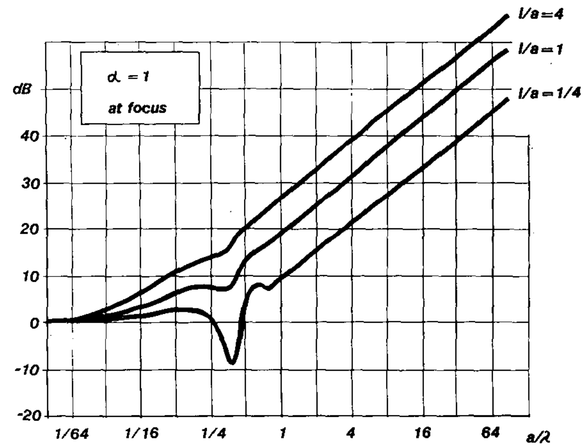

Fig. 5. Theoretical amplification at focus with different l/a ratios as a function of the ratio between focal distance and sound wavelength, a/λ.

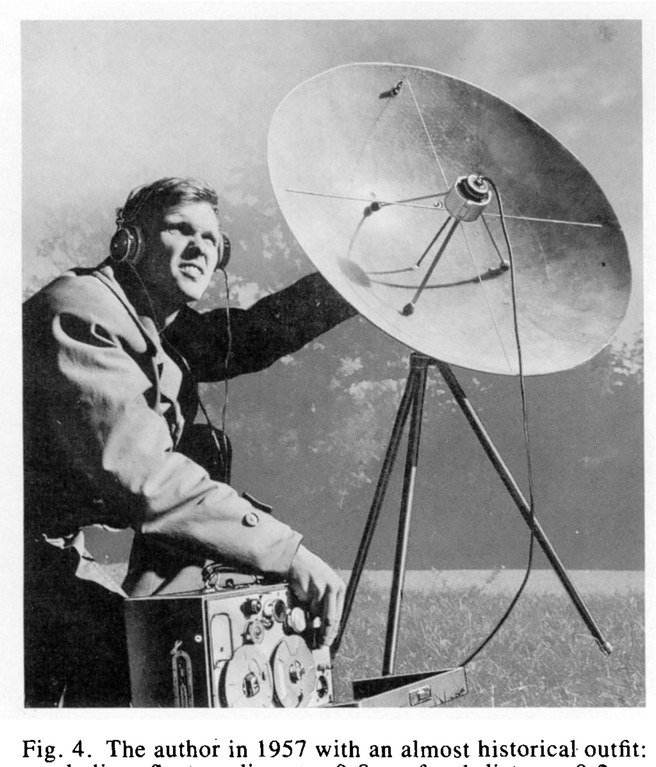

Fig. 4. The author in 1957 with an almost historical outfit: parabolic reflector, diameter 0.8 m, focal distance 0.2 m, microphone Western Electric 633 A, tape recorder Maihak MMK 1.

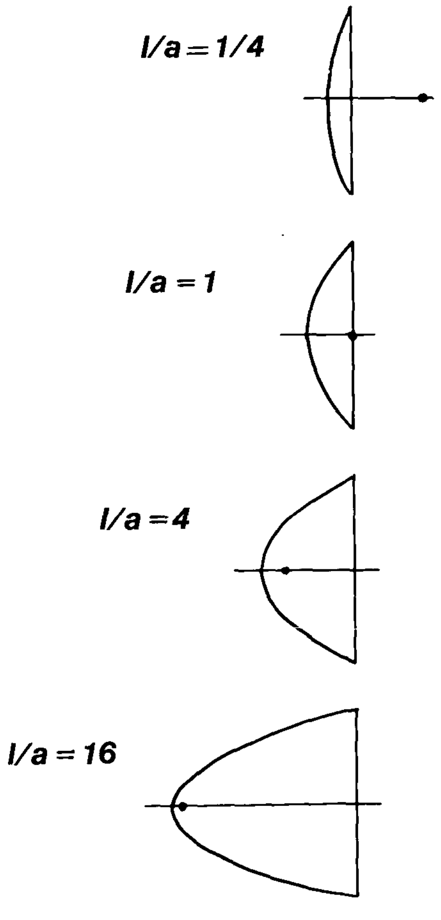

Fig. 6. Reflector shapes with different l/a ratios, but having the same diameter.

transport. The amplification for this shape is only 1.5 dB below maximum, a reasonable loss.

The amplification at points on axis inside and outside the focus is given by Eqs. (20) and (21) and is plotted in Figs. 9–11 for the three reflector shapes $l/a = ¼$, 1, and 4, respectively. For higher frequencies the amplifications deviate from the straight line. Then the sound waves reflected from different parts of the reflector no longer arrive at the focus in phase.

The same occurs at the focus when the sound source is located on axis but not at infinity, that is, for a diverging, not plane sound field, as will be seen later.

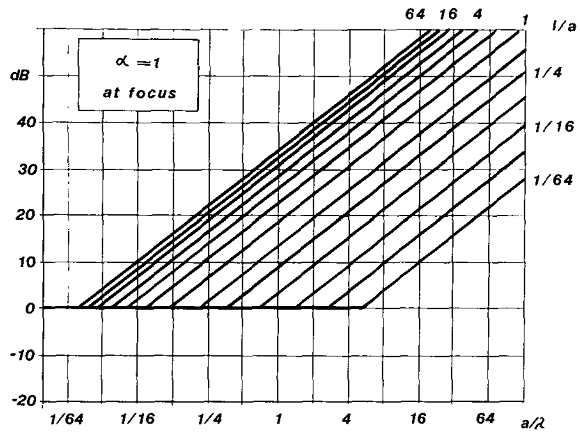

Fig. 7. Theoretical amplification at focus as a function of the ratio between focal distance and sound wavelength, a/λ, for reflectors with different l/a ratios, and omitting the direct sound.

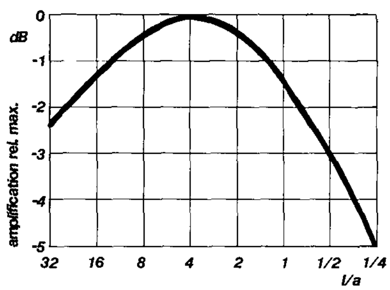

Fig. 8. Relative amplification for reflectors with different l/a ratios, but having the same diameter.

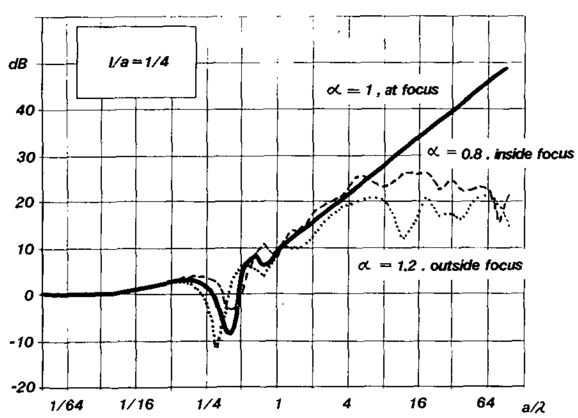

Fig. 9. Theoretical amplification on axis for a reflector with $l/a = ¼$.

The curves in the diagrams are for illustrative purposes only. For actual cases the formulas have to be used to carry out the calculations.

Finally the Appendix gives the expression for the directivity of the reflector [Eq. (22)]. Here practical measurements are of greater interest than the complicated theory.

3 PRACTICAL MEASUREMENTS OF THE ACOUSTICAL PROPERTIES OF THE PARABOLIC REFLECTOR

3.1 Amplification

Fig. 12 shows measurements carried out on a reflector with a diameter of 0.5 m and a focal distance of 0.125 m, which gives $l/a = 1$. As can be seen, there is good agreement with the theory. For frequencies higher than 4000 Hz we get lower values for amplification than predicted by theory. This is due to two factors. The measurements were carried out with a sound source at a distance of only 4.5 m in an anechoic chamber. This gives an effect compared with moving the focus forward, out from the reflector, and the effect can be equaled with the curve given in Fig. 10 for $\alpha = 1.2$, outside the focus. Furthermore, at 1000 Hz the sound wavelength is 34 mm, and at this frequency the microphone

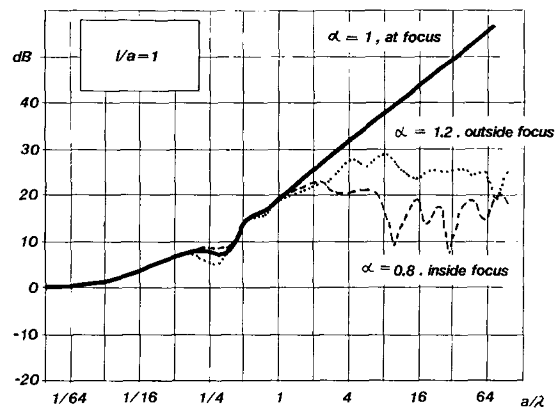

Fig. 10. Theoretical amplification on axis for a reflector with $l/a = 1$.

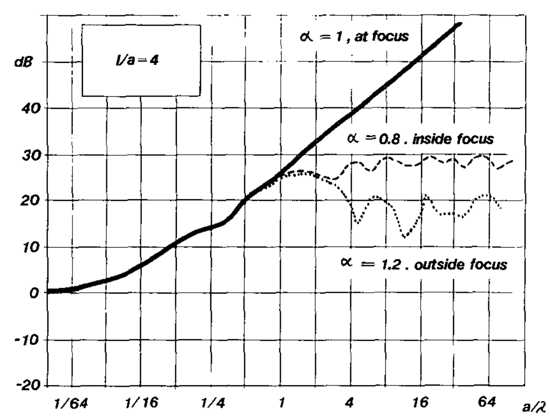

Fig. 11. Theoretical amplification on axis for a reflector with $l/a = 4$.

membrane is not exposed to the high amplification over its entire surface. This will affect all microphones that are large compared with the sound wavelength. The focal point of the reflector with a mathematically correct surface represents a strange sound field at high frequencies. This also explains why there is no intent to use a directional microphone together with a reflector, since it is desirable to pick up the sound from the part of the hemisphere represented by the reflecting surface.

Fig. 13 shows the measurements published by Kellogg in 1938. Kellogg himself was rather suspicious of them (personal communication), but they conform astonishingly well with the theory. In fact they conform even better than measurements carried out much later [9], as shown in Fig. 14. Although not published, the theoretical explanations given in [9] are widely circulated among biological researchers. The results are not correct since they do not fully take into account the mathematics of the parabola.

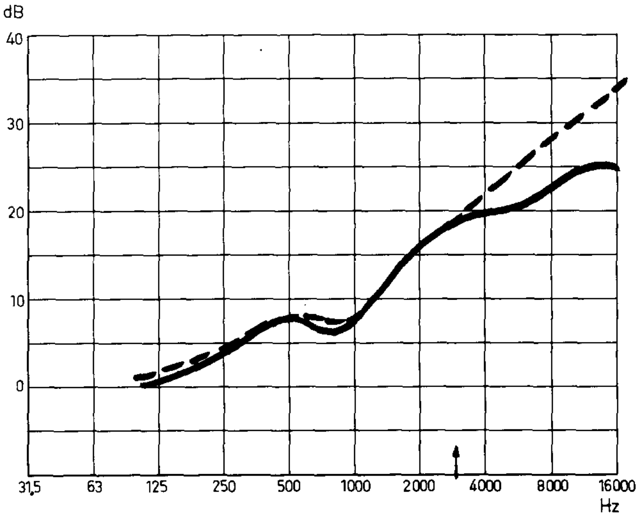

Fig. 12. Measured amplification of a reflector with diameter 0.5 m, a = 0.125 m, l/a = 1, and a/λ = 1 for f = 2820 Hz. ---- theory; ——measurement (see Ref. [10]), with a 0.5-in (12-mm) condenser microphone, Brüel & Kjaer. Distance to sound source 4.5 m.

3.2 Directivity

The directivity of the parabolic reflector has been measured in an anechoic chamber, and the results are given in Fig. 15. The values are equalized to 0 dB on axis. Assuming a tolerance of −5 dB at 8 kHz, relative to 0 dB on axis, the opening of the reflector is approximately ±5°. For a reflector twice this size, the same will occur at 4 kHz already. The shielding effect of the reflector is clearly visible at higher frequencies. The increase at 180° is due to the diffraction of sound waves on the edge of the reflector and the waves arriving at the focus in phase and thus adding up the sound pressure. The measurements were made with a Sennheiser MD 21 dynamic microphone and are only valid for this particular reflector–microphone combination.

4 PRACTICAL USE OF THE PARABOLIC REFLECTOR FOR WILDLIFE SOUND STUDIES

The parabolic reflector has a most obvious amplification. The function is compared with open microphones and shown schematically in Fig. 16. The disadvantage, however, is its frequency dependence. The widespread use of the reflector for wildlife sound recording can be explained, as Kellogg already did, by the limited frequency range of many animal voices. Many birds, for example, have their main sound energy distributed within one octave, and a 6-dB change in frequency response is of minor importance when listening to it. For scientific analysis it is necessary to take this into account, and with a simple RC filter it is easy to compensate for this effect.

A simple example of the use of the reflector is shown in Figs. 17–19. Fig. 17 is the careful analysis of a recording of a bird in a Stockholm park. The recording

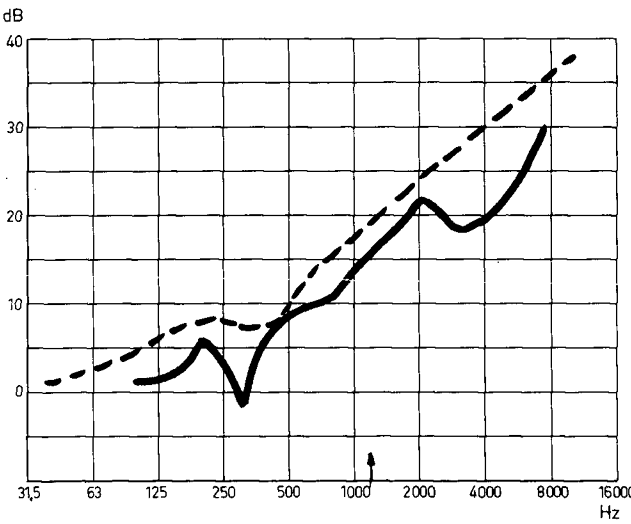

Fig. 13. Measured amplification of a reflector with diameter 0.81m, a not given but probably 0.305 m, and a/λ = 1 for f = 1113 Hz. ---- theory; —— measurement (see Ref. [6]). Distance to sound source 60 ft (≈20 m).

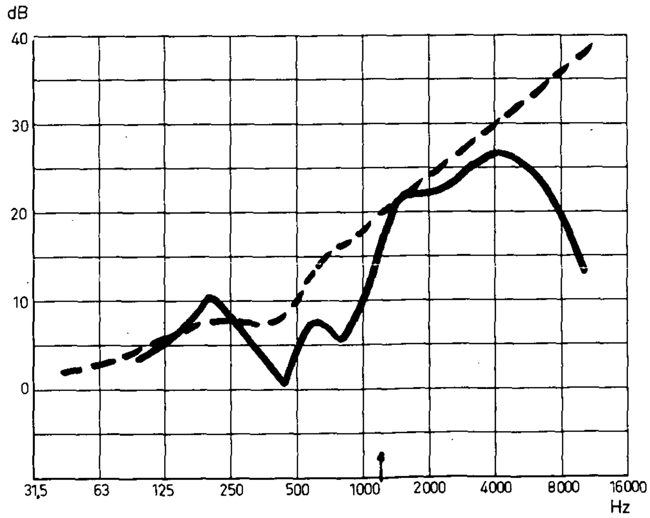

Fig. 14. Measured amplification of a reflector with diameter 0.914 m, a = 0.305 m, and a/λ = 1 for f = 1113 Hz. ---- theory; ——measurement (see Ref. [9]).

has a very poor signal-to-noise ratio between the bird's song and the traffic noise. The schematic result of a recording made with a reflector and with the bird at a distance of 25 m, which is a more common distance, is shown in Fig. 18. Without the reflector the recording would have been dominated by the traffic noise. Processing the recording with a filter to compensate for the reflector, −6 dB per octave, and to sort out the traffic noise, −18 dB per octave, will increase the signal-to-noise ratio dramatically (Fig. 19).

A further effect of the reflector is the shielding of the microphone from statistically arriving noise, the background noise (Fig. 20). This effect will depend on the shape of the reflector and is shown for three different cases assuming frequencies with wavelengths smaller than the diameter of the reflector.

For educational purposes it is necessary that in a recording the sound to be pointed out be lifted far enough above the background for the listener to know what sound is to be singled out. This means about 20 dB above the background chorus, and this is just what the parabolic reflector can do. Furthermore it increases the signal-to-noise ratio to an acceptable level.

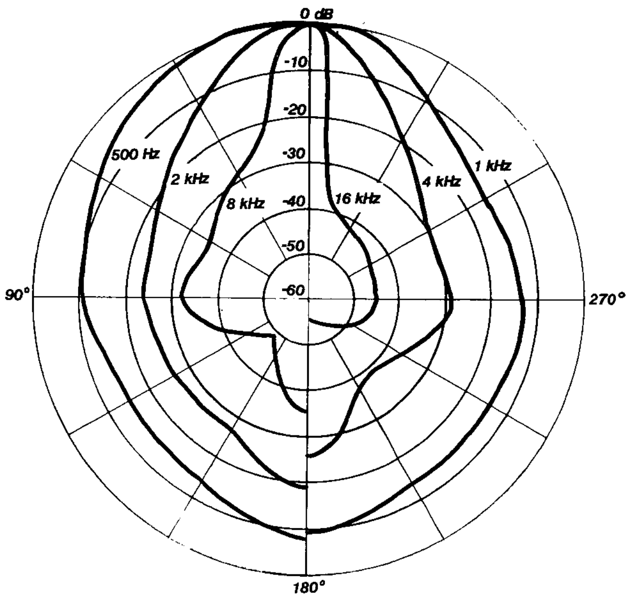

Fig. 15. Measured directivity of a reflector with a dynamic microphone, Sennheiser MD 21, at focus, for different frequencies. Diameter 0.5 m; a = 0.125 m; l/a = 1; distance to sound source 4.5 m, in an anechoic chamber (see Ref. [10]).

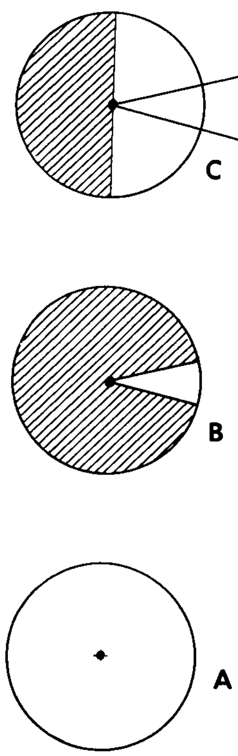

Fig. 16. Effect of parabolic reflector C compared with an open, pressure microphone A and a directional microphone B. A picks up sounds from all directions, B suppresses sounds from unwanted directions, and C amplifies wanted sounds from a selected direction.

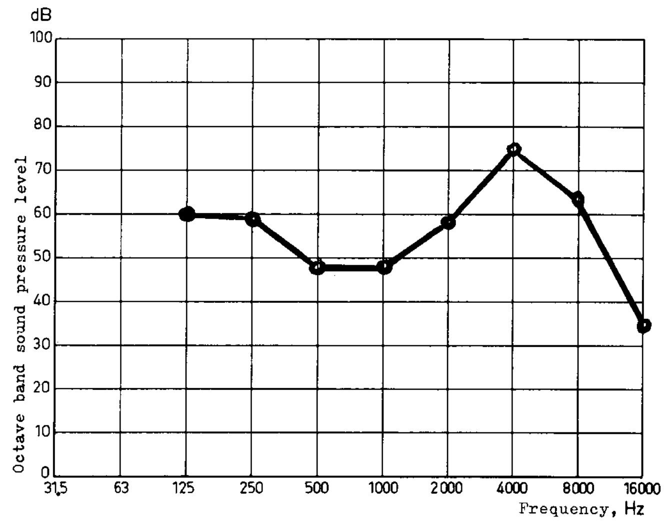

Fig. 17. Recording of a Chaffinch, *Fringilla coelebs*, the most common Swedish bird, in a Stockholm park. Recorded with precision instruments on April 2, 1957 at 11 A.M. Distance 3 m; traffic noise from town dominates below 1000 Hz.

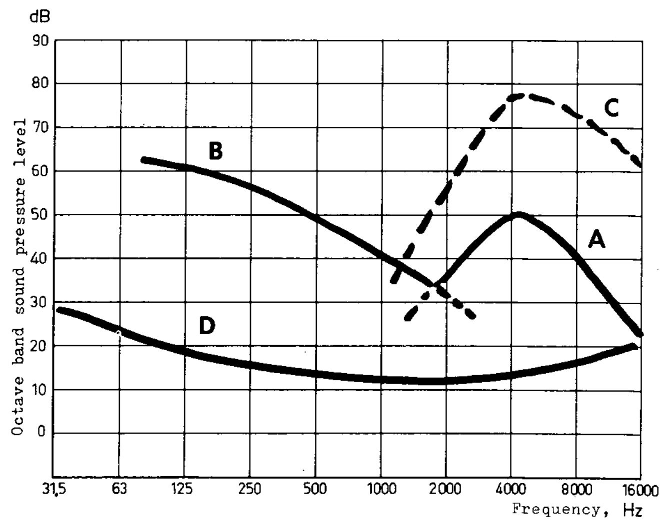

Fig. 18. A—Chaffinch, calculated for a distance of 25 m; B—traffic noise of town; C—Chaffinch, recorded with a parabolic reflector, diameter 0.8 m; D—electronic noise of recording equipment.

5 MATERIALS FOR REFLECTORS

From the acoustical point of view the requirements on the material of the reflector are not especially high. This can be easily understood by a simple calculation.

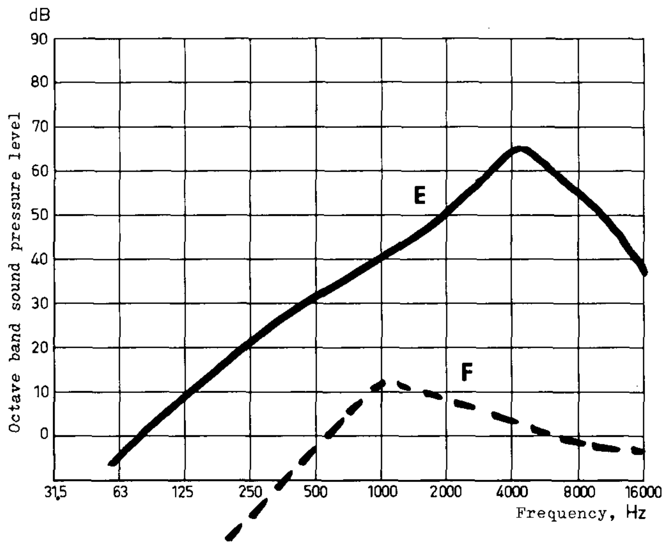

Fig. 19. E—recording of Fig. 18, processed with a filter, −6 dB per octave above 1000 Hz to compensate for the reflector response and −18 dB per octave below 1000 Hz to filter out traffic noise. F—electronic noise.

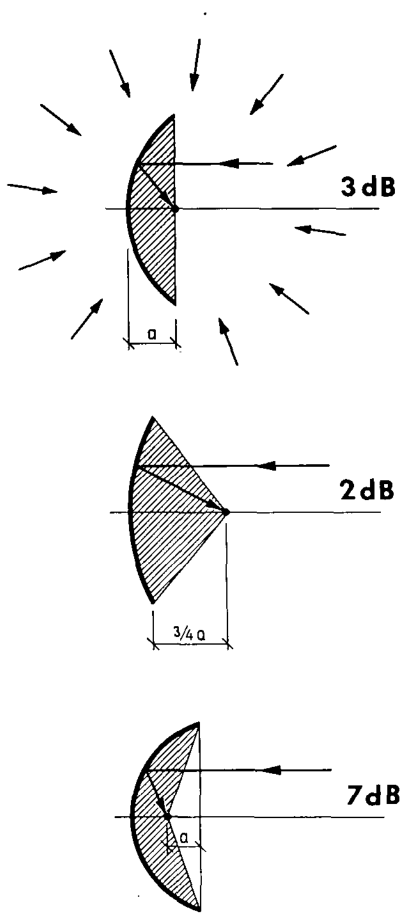

Fig. 20. Further effect of the reflector. The surface shields off the microphone from unwanted sounds to different degrees depending on the reflector shape. If the unwanted sounds are equally distributed in a sphere, the discrimination is 3, 2, and 7 dB, respectively. This effect assumes 2 >> D.

An absorption factor of, say, 0.4 means that the rest, 60%, is reflected. The loss of energy is consequently just 2 dB for such a material. If a reflector is made of this material, the loss is negligible compared with the amplification of the reflector.

Reflectors are usually made of 2-mm-thick glassfiber plastic, which is quite sufficient for adequate sound reflection. But even thinner plastic materials reflect enough to be usable for reflectors. In fact, reflectors made in the same shape as plastic umbrellas have been used with good results. It is more a question of finding umbrellas with the right parabolic shapes.

The use of the reflector for wildlife sound recordings is full of practical challenges, such as tripods for noiseless panoramic movements, microphone mountings, protection against wind, and so on, but these questions lie beyond the scope of this paper and are well covered in practical handbooks and other literature.

6 STEREOPHONIC RECORDINGS WITH A PARABOLIC REFLECTOR

It is even possible to adopt a reflector for stereophonic recordings by a simple method [10]. When a shield is put vertically in the center of the reflector, the reflector is divided into two symmetrical parts. One microphone is placed on each side of the shield as close to the focus as possible (Fig. 21). The shield separates the front into two sides and will thus create a stereo background when the sound is picked up by the two separate microphones. The sound source in the center of the sound picture will be equally amplified by the reflector to the microphones. This is a fair way of producing a recording of a soloist to a stereo background. A disadvantage is that the only way of balancing the soloist to the background is to change the distance from the soloist since this balance cannot be made electronically. The alternative would be to use normal recording techniques with A–B or M–S microphone systems. But these are very fragile systems for field work.

7 REFERENCES

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APPENDIX

THEORETICAL DERIVATIONS OF THE AMPLIFICATION AT VARIOUS POINTS

A.1 Definitions

$\phi(x, y, z, t)$	= velocity potential
$\bar{v}$	= particle velocity; = grad ϕ
p	= sound pressure
Z_s	= acoustic impedance
ρ	= density
c	= sound velocity
λ	= wavelength
f	= frequency; $f \cdot \lambda = c$
ω	= angular velocity; $= 2\pi f$
k	= wave number; $= 2\pi/\lambda$
(n, x)	= angle between normal and x axis
$F_p = \lvert p_1/p_2 \rvert$	= sound pressure amplification factor, pressure factor
$F_v = \lvert v_1/v_2 \rvert$	= particle velocity amplification factor, velocity factor
ln	= natural logarithm
lg	= Briggs' (base 10) logarithm

A.2 Mathematical Relations Governing the Parabola and the Paraboloid of Rotation

The parabola is the geometrical locus of a point whose distance from a given point is equal to its distance from a given straight line (Fig. 22). If a parabola is caused to rotate around its axis, it describes a surface called a paraboloid (of rotation).

Two properties of the paraboloid are fundamental to its application as an optical as well as an acoustic reflector (Fig. 23):

1) The angle between OP and the tangent through P equals the angle between FP and the same tangent.

2) For a given line at right angles to the axis the sum of OP and PF is constant.

The following definitions are used for the dimensions of the paraboloid (Fig. 24):

a = distance of vertex from focus
l = distance of vertex from plane of mouth
R = radius of mouth
D = diameter of mouth, in particular, if $l = a$, $D = D_a$; D_a is also known as focal chord.

A.3 Amplification at Focus

A rigid, infinitely thin disk's influence on a sound field can be calculated in the following way. At the surface the normal component of the particle velocity must be zero, which is satisfied by superposition of two sound fields, one from the incident sound wave and one arising when the disk moves with a particle velocity normal to the disk, which is the same as the velocity of the incident sound wave but in opposite phase.

The velocity potential of an infinitely thin disk vibrating in a direction normal to it (Fig. 25) is given by

$$\phi = -\frac{1}{2\pi} \int\int \frac{\partial\left(\frac{e^{-jkr}}{r}\right)}{\partial n} \phi_s dS \qquad (1)$$

where ϕ_s is the velocity potential of the disk surface.

For a bounded disk the value of ϕ_s at the edge of the disk is indeterminate. Eq. (1) can, however, be solved approximately. The effect of the rear side of the disk can be ignored at points near its center. Each side of the disk is assumed to radiate only into the hemisphere corresponding to it. The velocity potential of the sound

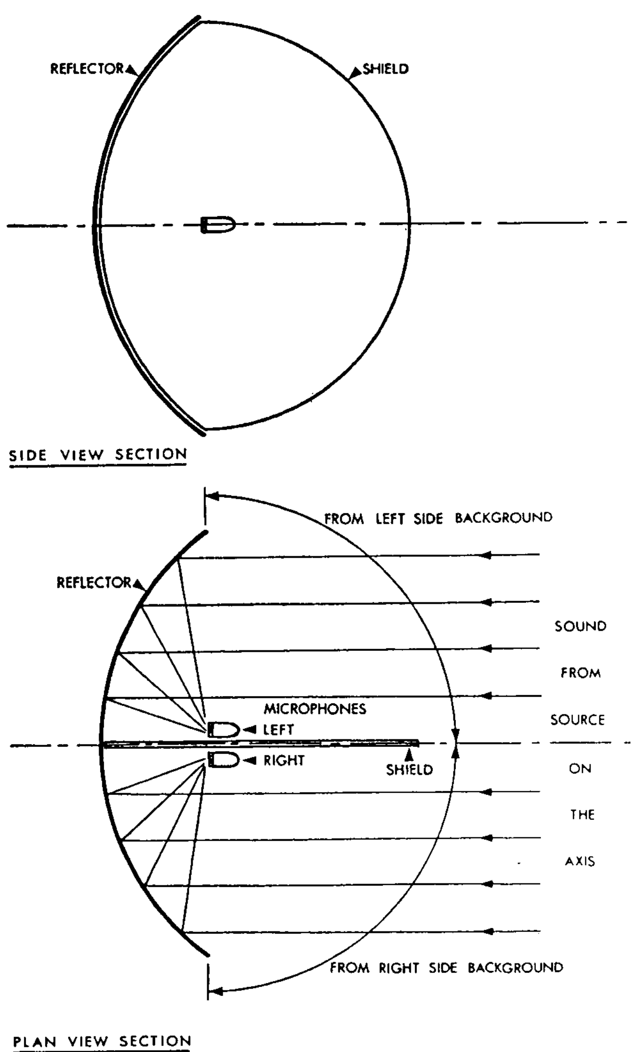


Fig. 21. A stereo parabola is constructed with a shield dividing the reflector into two symmetrical parts. The positions of the microphones are diagrammatic and would in reality be closer to the axis, and the sound waves would all be reflected toward a single focal point on this axis (see Ref. [10]).

wave is determined from

$$\phi = -\frac{1}{2\pi}\int\int \frac{\partial \phi_s}{\partial n}\frac{e^{-jkr}}{r}\,dS\ . \qquad (2)$$

Here the velocity of the disk in the direction of the normal $\partial\phi_s/\partial n$ is known. The direction of the normal is that of the incident sound wave. The velocity of the disk is of the same magnitude but in opposite phase to the velocity component of the sound wave in the normal direction.

Experimental measurements have proved this approximation to be acceptable. It can be assumed that a paraboloid should produce a still better separation between its concave and its convex surfaces, especially for points near the vertex and the focus.

A plane sound wave with velocity potential $\phi_i(x, y, z, t)$ parallel to the axis is thus assumed to strike the concave surface of the paraboloid. On the surface of the paraboloid the resulting sound field, that is, the sum of the incident and reflected fields, must be such that the normal component of the particle velocity on the surface of the paraboloid is zero.

The reflected sound wave can be treated as produced by a movement of the paraboloid with the velocity $\partial\phi_i(x, y, z, t)/\partial n$ in the normal direction and with the phase opposite to that of the incident sound wave at the same point. Here, too, account is taken of radiation from only the concave, reflecting side of the paraboloid. This approximation is good down to low frequencies where, however, relatively little sound energy is reflected.

The velocity potential ϕ_r of the reflected sound wave is obtained from

$$\phi_r(x, y, z, t) = \frac{1}{2\pi}\int\int \frac{\partial\phi_i(x, y, z, t)}{\partial n}\frac{e^{-jKr}}{r}\,dS\ . \qquad (3)$$

Secondary reflections are ignored. This approximation

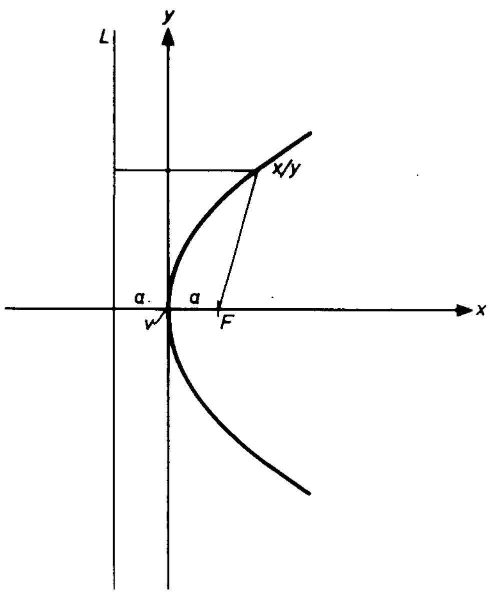


Fig. 22. Geometry of parabola. By definition the distance from x/y to straight line L equals the distance from x/y to focus F. a—focal distance; v—vertex; x axis—axis of parabola.

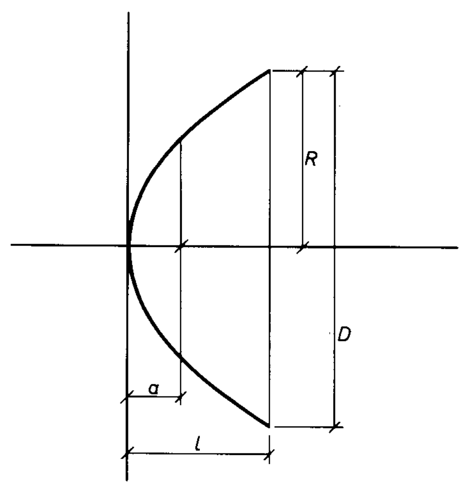


Fig. 24. Physical dimensions of paraboloid: a—distance of vertex from focus; l—distance of vertex from plane of mouth; R—radius of mouth; D—diameter of mouth.

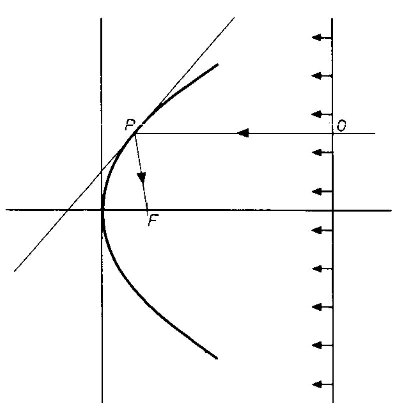


Fig. 23. A ray O is reflected by the surface of the paraboloid to focus F.

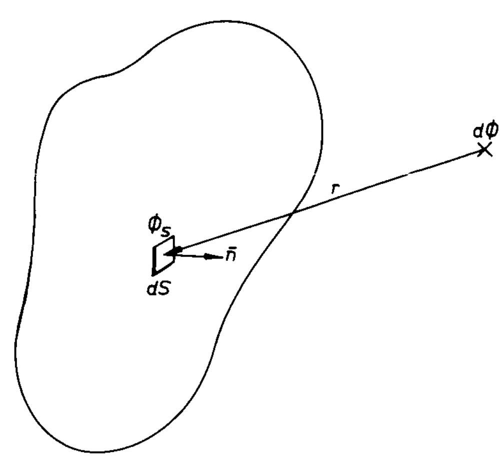


Fig. 25. Velocity potential $d\phi$ at distance of r from an infinitely thin disk dS, vibrating in a direction normal to it, $\bar{n}$.

has proved valid for both flat and deep reflectors and for high frequencies.

A.4 Sound Pressure at the Focus from an Incident Sound Wave Parallel to the Axis

The velocity potential of the incident wave is

$$\phi_i = Ae^{j(\omega t + Kx)} \tag{4}$$

Note that the wave moves in the negative x direction. Omit the term $Ae^{j\omega t}$ so that $\phi_i = Ae^{jKx}$. The normal component of the particle velocity on the surface of the paraboloid is

$$-\frac{\partial \phi_i}{\partial n} = -\frac{\partial \phi_i}{\partial x} \cos(n, x)$$

$$= -jKe^{jKx} \cos(n, x) . \tag{5}$$

The velocity potential of the reflected wave is calculated according to

$$\phi_r = \frac{jK}{2\pi} \int\int \frac{e^{jKx}e^{-jKr}}{r} \cos(n, x)\, dS . \tag{6}$$

If the value at the focus is inserted, $r = x + a$, we get

$$\phi_r = \frac{jKe^{-jKa}}{2\pi} \int\int \frac{\cos(n, x)}{x + a} dS . \tag{7}$$

For integration $dS = 2\pi y\, ds$, where ds is an element of the parabola arc,

$$dS \cos(n, x) = 2\pi y\, ds \cos(n, x) = 2\pi y\, dy .$$

From Sec. A.2 we obtain $y\, dy = 2a\, dx$ and

$$\phi_r = \frac{jke^{-jka}}{2\pi} \int_0^l \frac{4\pi\, dx}{x + a}$$

$$= j2ake^{-jka} \ln\left(1 + \frac{l}{a}\right) \tag{8}$$

$$\phi_r = j2ake^{-jka} \ln\left(1 + \frac{R^2}{4a^2}\right) . \tag{9}$$

The velocity potential of the sound field at the focus is

$$\phi_a = \phi_i + \phi_r$$

$$= e^{jka} + j2kae^{-jka} \ln\left(1 + \frac{l}{a}\right) . \tag{10}$$

The sound pressure at the focus is

$$p_a = -\rho \frac{\partial \phi}{\partial t}$$

$$= j\omega\rho(\phi_i + \phi_r) \qquad \text{quasistatic} \tag{11}$$

$$\frac{p_a}{p_i} = \frac{j\omega\rho(\phi_i + \phi_r)}{j\omega\rho\phi_i}$$

$$= 1 + j2ake^{-j2ak} \ln\left(1 + \frac{l}{a}\right) . \tag{12}$$

The pressure factor at the focus is

$$F_p = \left|\frac{p_a}{p_i}\right| \tag{13}$$

$$F_p = \left\{1 + \left[4\pi \frac{a}{\lambda} \ln\left(1 + \frac{l}{a}\right)\right]^2 + 8\pi \frac{a}{\lambda} \ln\left(1 + \frac{l}{a}\right) \sin 4\pi \frac{a}{\lambda}\right\}^{1/2} \tag{14}$$

When $\sin 4\pi\, a/\lambda = \pm 1$, we get

$$F_p = 4\pi \frac{a}{\lambda} \ln\left(1 + \frac{l}{a}\right) \pm 1 . \tag{15}$$

If the amplification is high, the expression can be simplified to

$$F_p \approx 4\pi \frac{a}{\lambda} \ln\left(1 + \frac{l}{a}\right)$$

$$= 4\pi \frac{a}{\lambda} \ln\left(1 + \frac{R^2}{4a^2}\right) . \tag{16}$$

This expression is also obtained if the incident wave is neglected compared with the reflected wave.

The amplification can also be derived by the reciprocity theorem: A sound source at the focus will give the same field at a far point on axis as the same sound source at the same far point will produce at the focus.

During this derivation it is shown that a sound source at the focus gives a plane circular radiation source in the plane of the mouth with the particle velocity being a function of the distance from the axis,

$$v = \frac{v_0}{1 + y^2/4a^2} . \tag{17}$$

If instead the particle velocity at the mouth is assumed to be constant, the factor $\ln(1 + l/a)$ in Eq. (12) is replaced by $(l/a)/\sqrt{1 + l/a}$. These two expressions are compared in Table 1. As can be seen, there is no significant difference for practical values of $1/a$.

It can be shown that the maximum amplification occurs for $R = 3.96a \approx 4a$, which gives

$$F_{p\ max} = \frac{\pi R}{\lambda} \ln 5 \approx 5 \frac{R}{\lambda} . \quad (18)$$

Finally the amplification in decibels, ΔL_p, is given by

$$\Delta L_p = 20 \log F_p \quad \text{dB} . \quad (19)$$

A.5 Sound Pressure on Axis from an Incident Plane Sound Wave Parallel to the Axis

The expressions for amplification on axis are more complicated. Here only the results of the calculations are given.

1) For points $0 < x_0 < a$,

$$\frac{p_x}{p_i} = 1 + j2ake^{-j2ak}[\text{Ci}(t_1) - \text{Ci}(t_2) + j\,\text{Si}(t_1) - j\,\text{Si}(t_2)] \quad (20)$$

where

$$t_1 = 2k(a - x_0)$$

$$t_2 = \frac{k(4a^2 - 4ax_0)}{\sqrt{(x_0 - l)^2 + 4al} + (2a - x_0 + l)} .$$

2) For points $x_0 > a$,

$$\frac{p_x}{p_i} = 1 + j2ake^{-j2ak}[\text{Ci}(t_1) - \text{Ci}(t_2) - j\,\text{Si}(t_1) + j\,\text{Si}(t_2)] \quad (21)$$

where

$$t_1 = 2k(x_0 - a)$$

$$t_2 = \frac{k(4ax_0 - 4a^2)}{\sqrt{(x_0 - l)^2 + 4al} + (2a - x_0 + l)} .$$

Table 1.

$\frac{l}{a}$	$\ln\left(1 + \frac{l}{a}\right)$	$\frac{l/a}{\sqrt{1 + l/a}}$
¼	0.223	0.224
½	0.405	0.408
1	0.693	0.707
2	1.099	1.155
4	1.609	1.789
8	2.197	2.667

In these expressions,

$$\text{Si}(t) = \int_0^t \frac{\sin \zeta}{\zeta} d\zeta$$

is the sinus integral and

$$\text{Ci}(t) = -\int_t^\infty \frac{\cos \zeta}{\zeta} d\zeta$$

is the cosinus integral (Fig. 26).

When $x_0 \to a$, both Eqs. (20) and (21) will give the same expression as shown earlier for amplification at the focus.

A.6 Directional Effects of Paraboloid

A complete calculation of the directivity gives complicated expressions, which are not shown here. A somewhat simpler expression is obtained by using the reciprocity theorem. This expression is valid for higher frequencies where directivity is also rather high.

$$\frac{p_a}{p_i} = 1 + j4ake^{-jka(1+\cos\alpha)}e^{-j2ak\sin(2\alpha/2)} \times \int_0^{R/2a} \frac{J_0(2akt \sin \alpha)}{1 + t^2} t\, dt . \quad (22)$$

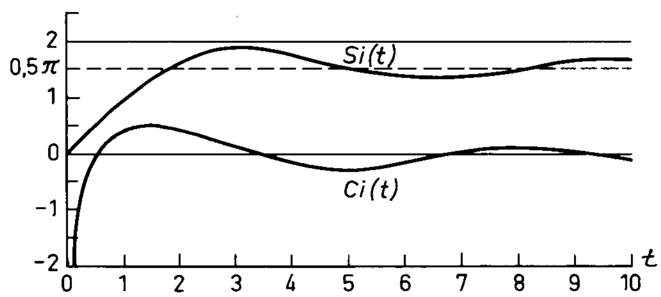


Fig. 26. Graphic representation of the expressions

$$\text{Si}(t) = \int_0^t \frac{\sin \xi}{\xi} d\xi \qquad \text{Ci}(t) = -\int_t^\infty \frac{\cos \xi}{\xi} d\xi .$$

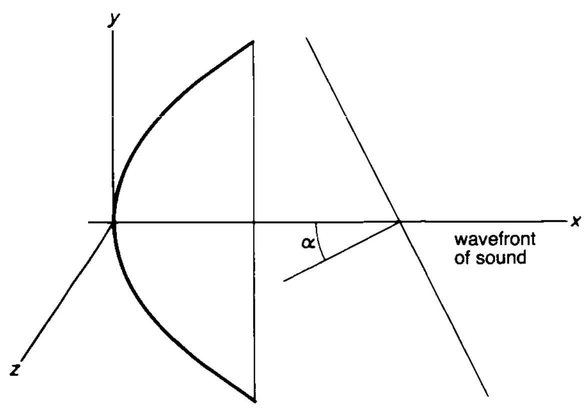


Fig. 27. Plane sound wave impinging on paraboloid with angle α from axis.

α is defined in Fig. 27 and $J_0(x)$ is the Bessel function of zero order.

A.7 Particle Velocity on Axis of the Paraboloid

When a plane sound wave is propagated parallel to the axis of the paraboloid, the direction of the velocity is also along the axis,

$$v = -\frac{\partial \phi}{\partial x_0} = -\frac{\partial(\phi_i + \phi_r)}{\partial x_0} .$$

The velocity at the focus becomes

$$v_a = -jke^{jka}\left\{1 - j2ake^{-j2ak}\left[\frac{2l/a}{1 + l/a} - \ln\left(1 + \frac{l}{a}\right) - j\frac{l}{a^2k(1 + l/a)^2}\right]\right\} . \tag{23}$$

The velocity factor is

$$F_v = \frac{v_a}{v_i} = 1 + j2ake^{-j2ak}\left[\ln\left(1 + \frac{l}{a}\right) - \frac{2l/a}{1 + l/a} + j\frac{l}{a^2k(1 + l/a)^2}\right] . \tag{24}$$

The term $jl/a^2k(1 + l/a)^2$ affects the result only at low frequencies.

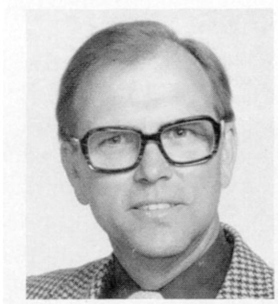
For the reflected wave the ratio between pressure and velocity at focus, such as the acoustical impedance, is given by

$$Z_{S_r} = \frac{p_r}{v_r} = \frac{\rho\, \partial\phi_r/\partial t}{-\partial\phi_r/\partial x_0} = \rho c \frac{\ln(1 + l/a)}{\frac{2l/a}{1 + l/a} - \ln(1 + l/a) - j\frac{l}{a^2k(1 + l/a)^2}} . \tag{25}$$

For a plane sound wave the corresponding value is $Z_s = \rho c$.

The sound field in the vicinity of the paraboloid thus departs most considerably from that of the plane wave.

THE AUTHOR



Sten Wahlström received an M.S. in electrical engineering from the Royal Institute of Technology in Stockholm, where he held a position as teacher and researcher within the field of acoustics from 1957–1973. He was then in charge of the environmental problems in Stockholm as the head of the technical department of the Environmental and Public Health Administration. He is now producer of scientific programs at the Swedish Radio.

Mr. Wahlström has taken part in the international acoustical standardization as a member of the corresponding Swedish national committee of ISO and has also been consultant to the Swedish Traffic Noise Committee. He was one of the founders of the Swedish Section of the AES and a former chairman. He has published several gramophone records and tapes with wildlife sounds, among them the world's first disk with stereo recordings of birds.